The Binomial Model and Risk Neutrality: Some Important Details

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Abstract

This paper reexamines the relationship between investors’ preferences and the binomial option pricing model of Cox, Ross, and Rubinstein (CRR). It is shown that the independence of the binomial option pricing model from investors’ preferences is a result of a special choice of binomial parameters made by CRR. For a more general choice of binomial parameters, risk neutrality cannot be obtained in discrete time. This analysis reveals the essential difference between the “risk neutral” valuation approach of Cox and Ross and the equivalent martingale approach of Harrison and Kreps in a discrete time framework.

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Introduction

Over a decade ago, the seminal work of Cox, Ross, and Rubinstein (CRR) [5] allowed the use of elementary mathematics in discrete time for option valuation. Since then, the binomial model has been applied and extended in many ways. In general, the binomial model has made three important contributions to the option literature. First, the binomial model has a considerable pedagogical value in demonstrating the economic intuition behind the formation of an arbitrage-free hedge portfolio for option pricing. Second, the binomial model allows simple continuous time numerical approximations of complex option valuation problems where no analytical closed form solutions exist. Finally, the binomial model demonstrates how option pricing can be done without any knowledge of the subjective preferences of the investors.

Thus, according to CRR, the discrete time binomial model is consistent with the risk neutrality argument of Cox and Ross [4]. This paper reexamines the consistency of the binomial option pricing model with the risk neutrality argument of Cox and Ross [4]. It is shown that risk-neutrality in discrete time is a consequence of a specific choice of binomial parameters by CRR. For a more general choice of binomial parameters (such as Jarrow and Rudd [7]) the discrete time binomial model is consistent with the equivalent martingale approach of Harrison and Kreps [8], but not with the risk-neutrality approach of Cox and Ross [4]. (The two approaches become consistent in the continuous time limit of the binomial model.)

The above observation underscores the necessity of imposing stronger restrictions on the asset return distributions in discrete time, in order to obtain risk neutrality.
Specifically, the binomial option pricing approach is consistent with many binomial stock price distributions, and only one out of these many distributions (the one given by CRR) is consistent with risk neutrality. Further, the above observation is also consistent with the discrete time contingent claims valuation models of Rubinstein [11], Brennan [3], and Stapleton and Subrahmanyam [12], which require that stronger distribution specific restrictions must be imposed on asset returns in order to obtain risk neutrality in the discrete time.\(^4\)

**A General Set of Binomial Parameters and Risk Neutrality**

This section begins with a review of the discrete time binomial model of CRR as follows. At time \(t = 0\), let a call option on a stock have \(T\) periods to expiration. Let \(T\) be divided into \(N\) number of sub-intervals. Let the current time be \(t = (T/N) \cdot (N-1)\). In other words, at the current time the call is one sub-interval or \((T/N)\) periods from the expiration date. At this time let the stock price equal \(S\). In the next period only two states can occur with the upward movement for the stock given by \(US\) and the downward movement given by \(dS\) (where \(u \geq d\)). The probability of upward and downward movements are \(q\) and \(1 - q\), respectively. Let \(C\) be the current price of a call option with exercise price \(E\). As shown by CRR the call price at the current time can be given as:

\[
C = \left[ p_u \cdot C^u + p_d \cdot C^d \right] / r^b
\]

where:

\[
p_u = \frac{r^b - d}{u - d}, \quad p_d = \frac{u - r^b}{u - d},
\]

\[
C^u = Max[0, uS - E]
\]
\[ C^d = \text{Max}[0, dS - E] \]

and

\( r = 1 + \text{riskless rate over a single period.} \)

\( h = T/N \) periods

Now consider the binomial parameters specified by CRR as follows:

\[ u = \exp(\sigma \cdot (T / N)^{0.5}) \]

(2)

\[ d = \exp(-\sigma \cdot (T / N)^{0.5}) \]

(3)

\[ q = (1/2) \cdot [1 + (\mu / \sigma) \cdot (T / N)^{0.5}] \]

(4)

where \( \mu \) is the preference parameter and \( \sigma \) is the volatility parameter. Since by definition 0 < \( q < 1 \), it is implied that \(-\sigma \cdot (T/N)^{0.5} < \mu \cdot (T/N) < \sigma \cdot (T/N)^{0.5}\).

Obviously since \( u \) and \( d \) do not contain the preference parameter \( \mu \), the risk-neutral probabilities \( p_u \) and \( p_d \), the future call prices \( C^u \) and \( C^d \), and the current call price \( C \), are all independent of preferences. However, as shown by Jarrow and Rudd (JR) [7] and others, the above choice of binomial parameters is not unique. JR specify the following choice of binomial parameters for the call option price in equation (1):

\[ u = \exp[\mu \cdot (T / N + \sigma \cdot (T / N)^{0.5}] \]

(5)

\[ d = \exp[\mu \cdot (T / N) - \sigma \cdot (T / N)^{0.5}] \]

(6)

\[ q = (1/2) \]

(7)

JR argue in favor of the above parameters because the first three moments for the stock's log-return implied by the above parameters are consistent with the respective
moments of the lognormal process over every length of discrete sub-interval \((T/N)\). These moments can be given as the mean \(=\mu \cdot (T/N)\), variance \(=(\sigma^2 \cdot (T/N)\), and skewness = 0. However, the three moments of the log-return for the binomial process using the CRR parameters are not consistent with the corresponding moments of the lognormal process. This can be seen from the following definitions of the moments using CRR parameters: the mean \(=\mu \cdot (T/N)\), variance \(=\sigma^2 \cdot (T/N) - \mu^2 \cdot (T/N)^2\), and skewness \(=2\mu \cdot [\mu^2 \cdot (T/N)^3 - \sigma^2 \cdot (T/N)^2]\). The CRR parameters imply a non-zero level of skewness for log-returns in discrete time. The variance and skewness of the binomial process implied by CRR parameters converge to the variance and the skewness of the lognormal process only in the continuous time limit (since \((T/N)^2\) and \((T/N)^3\) become insignificant in comparison to \((T/N)\) as \(N\) tends to infinity).

It can be shown that the JR parameters are not consistent with risk neutral approach to option valuation in discrete time. Substituting Jarrow and Rudd's choice of \(u\) and \(d\) (given in equations (5) and (6)) in equation (1) implies that the risk-neutral probabilities \(p_u\) and \(p_d\), the future call prices \(C_u\) and \(C_d\), and the current call price \(C\), all depend on the preference parameter \(\mu\). Hence, the resulting risk-neutral probabilities and the call price are preference dependent.

Now consider a general form of binomial parameters given as follows:

\[
u = \exp[m \cdot (T / N) + \sigma \cdot (T / N)^{0.5}] \tag{8}\]

\[
d = \exp[m \cdot (T / N) + \sigma \cdot (T / N)^{0.5}] \tag{9}\]

\[
q = (1/2) \cdot [1 + (((\mu - m) / \sigma) \cdot (T / N)^{0.5})] \tag{10}\]

where \(-\sigma \cdot (T/W)^{0.5} < (\mu - m) \cdot (T/N) < \sigma \cdot (T/N)^{0.5}\) (since \(0 < q < 1\)).
Given the above choice of binomial parameters, the first three moments of the log-return for the binomial process over a discrete sub-interval \((T/N)\) can be given as:

\[
\text{mean} = \mu \cdot (T/N), \quad \text{variance} = \sigma^2 \cdot (T/N) - (\mu - m)^2 \cdot (T/N)^2, \quad \text{and skewness} = 2(\mu - m) \cdot [(\mu - m)^2 \cdot (T/N)^3 - \sigma^2 \cdot (T/N)^2].
\]

The terms \((T/N)^2\) and \((T/N)^3\) become insignificant in comparison to \((T/N)\) as \(N\) tends to infinity, and therefore the variance and skewness of the binomial process implied by the above parameters converge to the variance and skewness of the lognormal process in the continuous time limit.

The above choice of binomial parameters (see equations (8) and (9)) implies that the terms \(p_u, p_d, C^u, C^d,\) and \(C\) in equation (1) depend upon the parameter \(m.\) To obtain a unique option price, investors may disagree about the preference parameter \(\mu\) (and thus, disagree on the actual probabilities), but all investors must agree on the parameter \(m.\) For the CRR model, all investors agree that the parameter \(m\) equals zero. For the JR model, all investors agree that the parameter \(m\) equals \(\mu.\) In general, if the parameter \(m\) cannot be uniquely determined, the resulting binomial option prices are not uniquely defined in discrete time.

To preclude arbitrage, additional restrictions must be imposed on the binomial parameters. Specifically, a unique equivalent probability measure must exist such that the stock price discounted at the riskless rate is a martingale with respect to this measure (see Harrison and Kreps [8]). To satisfy the above condition, the risk neutral probabilities \(p_u\) and \(p_d\) must be greater than zero in equation (1). This implies that for the CRR choice of parameters \(\exp(-\sigma \cdot (T/N)^5) < r^h < \exp(\sigma \cdot (T/N)^5),\) for the JR parameters \(\exp[\mu \cdot (T/N) - \sigma \cdot (T/N)^5] < r^h < \exp[\mu \cdot (T/N) + \sigma \cdot (T/N)^5],\) and for the revealed preference parameters \(\exp[m \cdot (T/N) - \sigma \cdot (T/N)^5] < r^h < \exp[m \cdot (T/N) + \sigma \cdot (T/N)^5].\)
Two limitations of the discrete time binomial approach can now be summarized with respect to the general binomial parameters given in equations (8), (9), and (10), as follows:

1. If \( m = \mu \), then the binomial model is preference dependent and is inconsistent with the risk-neutrality argument of Cox and Ross [4] in discrete time.

2. If \( m \neq \mu \), then risk neutrality can still be obtained since investors are allowed to disagree about the preference parameter \( \mu \). However, the difficulty here is in determining a unique value for \( m \), either theoretically or empirically. Different values of \( m \) will result in different call option prices in the discrete time.

Fortunately, both the above limitations of the binomial approach can be resolved in the continuous time limit. It can be shown that the dependence of the discrete time binomial option pricing model on the parameter \( m \) diminishes as the number of sub-intervals \( N \) becomes large. With an infinitely large \( N \), the Black and Scholes [2] option formula is obtained as a limiting case of the binomial option formula (see the Appendix). Thus, the binomial model is independent of the parameter \( m \), only in the continuous time limit. This underscores the necessity of continuous time portfolio rebalancing to guarantee risk neutrality as originally noted by Black and Scholes [2] and subsequently formalized by Cox and Ross [4].

**Conclusions**

This paper shows that the consistency of the discrete time binomial option pricing model of CRR with the risk neutrality argument of Cox and Ross [4] depends upon a specific choice of binomial parameters. For a more general choice of binomial
parameters, the resulting option prices may be preference dependent. This preference
dependence diminishes as the number of sub-intervals $N$ becomes large and disappears
completely only in the continuous time limit as the binomial model converges to the
Black and Scholes model. The implications of these results pertain to one of the central
issues of modern finance: the risk neutral pricing of contingent claims. In order to obtain
risk neutrality, the CRR model must assume very specific behavior regarding the price
changes of underlying assets.

However, both theoretically and practically speaking, alternative price behavior
models are reasonable. This paper demonstrates a binomial option pricing model using an
alternative and reasonable specification of behavior in underlying asset price changes and
demonstrates that risk neutrality does not obtain in the model. Thus, this paper provides
an additional demonstration of the failure of risk neutrality to obtain in discrete time in
particular cases.
Appendix

This appendix shows that the binomial option pricing model converges to the Black-Scholes model for a general choice of binomial parameters given by equations (8), (9), and (10), as the number of sub-intervals $N$ becomes infinitely large. Though it is possible to demonstrate the actual convergence of the binomial call option price to the Black-Scholes call option price, a much easier and intuitive proof follows from Cox and Rubinstein [6], page 209. Using the alternative approach, the Black-Scholes P.D.E. is derived from the binomial equation for the general choice of binomial parameters.

Reconsider the call option defined in the second section at time $t$ ($0 \leq t \leq T$). The call price at time $t$ can be given as follows:

$$C_t = \left[ p_u \cdot C_{t+h}^u + p_d \cdot C_{t+h}^d \right] / r^h$$  \hspace{1cm} (A1)

where the parameters $p_u$, $p_d$, $r^h$, and $h$ are defined in equation (1), and the general binomial parameters $u$ and $d$ are defined in equations (8) and (9). Following Cox and Rubinstein, the call price at time $t$ is assumed to be a continuously differentiable function of the stock price at time $t$, and the time remaining to the expiration date. Thus, $C_t = C(S_t, T- t)$, $C_{t+h}^u = C(u \cdot S, T - (t + h))$, $C_{t+h}^d = C(d \cdot S_t, T - (t + h))$, where subscript $t$ implies the time $t$ price of a given security. By appropriate substitutions equation (A1) can be rewritten as:
\[ r^h - \exp[m \cdot h - \sigma \cdot (h^5)] \]  
\[ \frac{\exp[m \cdot h + \sigma \cdot (h^5)] - \exp[m \cdot h - \sigma \cdot (h^5)]}{\exp[m \cdot h + \sigma \cdot (h^5)] - \exp[m \cdot h - \sigma \cdot (h^5)]} \cdot C(\exp[m \cdot h + \sigma \cdot (h^5)] \cdot S_i, T - (t + h)) \]  
\[ + \frac{\exp[m \cdot h + \sigma \cdot (h^5)] - r^h}{\exp[m \cdot h + \sigma \cdot (h^5)] - \exp[m \cdot h - \sigma \cdot (h^5)]} \cdot C(\exp[m \cdot h - \sigma \cdot (h^5)] \cdot S_i, T - (t + h)) \]  
\[ - r^h \cdot C(S_i, T - t) = 0 \]  

(A2)

By Taylor series expansions of the expressions

\[ C(\exp[m \cdot h + \sigma \cdot (h^5)] \cdot S_i, T - (t + h)) \]  
and \[ C(\exp[m \cdot h - \sigma \cdot (h^5)] \cdot S_i, T - (t + h)) \] around the point \((S_b, T - t)\):

\[ C(\exp[m \cdot h + \sigma \cdot (h^5)] \cdot S_i, T - (t + h)) = C_i + [\exp[m \cdot h + \sigma \cdot (h^5)] - 1] \cdot S_i \cdot \frac{\partial C_i}{\partial S_i} \]  
\[ + \frac{1}{2} \cdot [\exp[m \cdot h + \sigma \cdot (h^5)] - 1] \cdot S_i \cdot \frac{\partial^2 C_i}{\partial S_i^2} + \frac{1}{2} \cdot [\exp[m \cdot h + \sigma \cdot (h^5)] - 1]^2 \cdot S_i^2 \cdot \frac{\partial^3 C_i}{\partial S_i^3} + h^2 \cdot \frac{\partial C_i}{\partial t} + ... \]  

(A3)

and a similar expression for \( C(\exp[m \cdot h - \sigma \cdot (h^5)] \cdot S_i, T - (t + h)) \), except \( -\sigma \cdot (h^5) \) replaces \( \sigma \cdot (h^5) \).

By Taylor series expansions of the expressions

\[ \exp[m \cdot h + \sigma \cdot (h^5)] \] and \[ \exp[m \cdot h - \sigma \cdot (h^5)] \]:

\[ \exp[m \cdot h + \sigma \cdot (h^5)] = 1 + [m \cdot h + \sigma \cdot (h^5)] + \frac{1}{2} \cdot [m \cdot h + \sigma \cdot (h^5)]^2 + ... \]  

(A4)

and a similar expression for \( \exp[m \cdot h - \sigma \cdot (h^5)] \), except \( -\sigma \cdot (h^5) \) replaces \( \sigma \cdot (h^5) \).

Finally, by a Taylor series expansion of \( r^h \):

\[ r^h = \exp(h \cdot \log r) = 1 + h \cdot \log r + \frac{1}{2} \cdot (h \cdot \log r)^2 + ... \]  

(A5)

By substitution of the appropriate values from equations (A3), (A4), and (A5) into equation (A2) and simplifying:
\[
\frac{1}{2} \cdot \sigma^2 \cdot S_i^2 \cdot \frac{\partial^2 C_i}{\partial S_i^2} \cdot h + (\log r) \cdot S \cdot \frac{\partial C_i}{\partial S_i} \cdot h + \frac{\partial C_i}{\partial t} \cdot h - (\log r) \cdot C_i \cdot h + Z = 0
\]  \hspace{1cm} (A6)

where \( Z \) contains all terms of higher orders of \( h \) (i.e. \( Z = \{\text{terms with } (h)^{1.5}, (h)^{2}, \ldots, \} \)).

Dividing equation (A6) by \( h \):

\[
\frac{1}{2} \cdot \sigma^2 \cdot S_i^2 \cdot \frac{\partial^2 C_i}{\partial S_i^2} + (\log r) \cdot S \cdot \frac{\partial C_i}{\partial D_i} + \frac{\partial C_i}{\partial t} - (\log r) \cdot C_i + \frac{Z}{h} = 0 \hspace{1cm} (A7)
\]

It can be seen from the above equation that the revealed preference parameter \( m \) (see equation (A2)) is contained only in the term \( Z/h \). For non-infinitesimal values of \( h \), the magnitude of \( Z/h \) will be significant, and the solution to equation (A7) will depend upon the revealed preference parameter. However, as the number of sub-intervals \( N \) tends to infinity, the term \( Z/h \) goes to zero (as \( h \) goes to zero), but other terms do not. Thus, in the continuous time limit, equation (A7) converges to the Black-Scholes partial differential equation, which is independent of the revealed preference parameter \( m \). Q.E.D.
End Notes

1. Recently, the economic intuition behind the binomial approach has been extended to a multivariate-multinomial approach by He [9] using the dynamic complete market framework. The arguments given in this paper apply to the multivariate-multinomial models, as the binomial model can be considered as a special case of these models.

2. See Merton [10], pp. 337-347, for the application of the risk-neutrality argument of Cox and Ross [4] to the binomial model of CRR.

3. Recently, Beck [1] demonstrated that the traditional derivation of Black-Scholes option formula is mathematically unsatisfactory. Specifically, Beck shows that the hedge portfolio used in Black-Scholes is neither riskless nor self-financing. Beck presents an alternative derivation of the Black-Scholes formula that avoids these inconsistencies. Though the issues analyzed in this paper are quite different, it is similar to the Beck’s paper in spirit.

4. For example, assume that investors preferences are given by CPRA utility. Then, it is only necessary that the underlying asset and the market portfolio be bivariate lognormally distributed to obtain risk-neutrality in the discrete-time. Of course, the above assumption is not necessary in the continuous-time analog of the above model (i.e., Black and Scholes [2]).

5. Jarrow and Rudd [7], pp. 179-190, show their binomial parameters to be consistent with risk-neutrality in the continuous time limit. This is consistent with the Appendix A of this paper, which obtains a similar result.

6. CRR do not provide any theoretical justification or empirical evidence for choosing zero as the appropriate value for $m$.

7. At the first glance, equation (A7) looks slightly different from the Black-Scholes partial differential equation. However, note that $r$ equals 1 plus the discrete-time single period riskless rate (see equation (1)). However, Black-Scholes use the continuous-time riskless rate, which can be defined as $R$. From the basic rules of compounding over time, the relationship between $r$ and $R$ over any length of time $h$ is given as $r^h = \exp(R \cdot h)$, which in turn implies $\log r = R$. 
References


